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A RESOURCE ALLOCATION MODEL  
WITH DEVELOPMENT COSTS AND PRIOR INVESTMENTS

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U.S. NAVAL ORDNANCE LABORATORY  
WHITE OAK, MARYLAND

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A Resource Allocation Model with Development Costs and Prior Investments

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Commander

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By direction

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I. INTRODUCTION

The theory of max-min is extended to a class of functions having direct application to the allocation of resources among strategic retaliatory systems. The model presented below permits the inclusion of both committed resources and "prices of admission."

The price of admission includes development costs, and these terms are used interchangeably; however, preliminary research costs necessary for the serious consideration of any product or system are excluded. By its nature, research allocation must be considered separately. Committed resources includes the value of existing systems and resources previously committed for a variety of (possibly irrational) reasons.

The problems considered describe "zero-sum" competition among two players. In the absence of development costs and committed resources the model becomes a zero-sum, two-person continuous game with a continuous payoff function. Specifically, we consider a payoff function for the x-player

$$(1.1) \quad F(x, y) = \sum_{i=1}^n f_i(x_i, y_i)$$

where  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n)$  and

$$(1.2) \quad f_i(x_i, y_i) \equiv \begin{cases} 0 & : 0 \leq x_i \leq q_i; \\ h_i(x_i) & : x_i \geq q_i, 0 \leq y_i < r_i; \\ g_i(x_i, y_i) & : x_i \geq q_i, y_i \geq r_i. \end{cases}$$

Call  $f_i(x_i, y_i)$  the value of the  $i$ th system;  $q_i$  denotes the x-player's price of admission for the  $i$ th system and  $r_i$  denotes the y-player's price of admission. Since a competitive situation is being described  $g_i(x_i, y_i)$  is an increasing (decreasing), concave (convex) function of  $x_i(y_i)$ . For each  $i$  define

$$h_i(x_i) \equiv g_i(x_i, r_i)$$

and

$$h_i(q_i) \equiv 0.$$

Further assume that  $g_i(x_i, y_i) > 0$  whenever  $x_i > q_i$ . The allocation of resources to the  $i$ th system is denoted  $x_i$  or  $y_i$ .

Allowing  $R_x$  and  $R_y$  to be the total resources (excluding research funds) available to the  $x$ -player and the  $y$ -player respectively, the allocation is constrained by

$$(1.3) \quad x_1 + \dots + x_n = R_x \quad ; \quad y_1 + \dots + y_n = R_y.$$

The allocation is additionally constrained by

$$(1.4) \quad x_i \geq s_i \geq 0 \quad ; \quad y_i \geq t_i \geq 0 \quad (i = 1, 2, \dots, n)$$

where  $s_i$  and  $t_i$  represent the amount of previously committed resources to the  $i$ th system. We assume for convenience of presentation that  $s_i > 0$  ( $t_i > 0$ ) implies  $s_i > q_i$  ( $t_i > r_i$ ). For physical reasons, we also assume  $R_x > \max_i q_i$ ,  $R_y > \max_i r_i$  and that  $t_i = 0$  whenever  $s_i = 0$ .

Since we are interested in applications where the  $x$ -player moves first, the purpose of the problem is to determine the "residual value"  $V$  and "optimal strategies"  $u = (u_1, \dots, u_n)$  and  $w = (w_1, \dots, w_n)$  such that

$$(1.5) \quad V = F(u, w) = \text{Max}_x P(x)$$

where

$$P(x) \equiv \text{Min}_y F(x, y)$$

$P(x)$  is called the security function.

A particular example is determined by

$$(1.6) \quad g_i(x_i, y_i) = v_i(x_i - q_i) \exp [-a_i(y_i - r_i)]$$

where  $v_i$  and  $a_i$  are constants. The zero development cost, zero committed resources game corresponding to this example has been solved by Danskin [1]. Shere and Cohen have extended Danskin's max-min theory to solve this example with committed resources but no development costs [2] and with development costs but no committed resources [3].

In the following section (1.1)-(1.5) is solved in a constructive manner; additional constraints are imposed as needed on  $g_i(x_i, y_i)$ . The final section consists of observations on problems related to (1.1)-(1.5) and conclusions.

## II. MATHEMATICAL ANALYSIS

This section is organized in two parts. The first part consists of nine lemmas and theorems which describe the nature of the solution of (1.1)-(1.5). It is shown by adding hypotheses related to the differentiability of  $g_i(x_i, y_i)$  that the optimal strategies  $(u, w)$  are also optimal strategies of a certain game. A scheme for determining the game and consequently  $(u, w)$  is subsequently provided.

The following four lemmas are trivial extensions of the corresponding lemmas of [3].

Lemma 1. Let  $x = (x_i)$  and  $\eta(x) = (\eta_i(x))$ . If  $P(x) = F(x, \eta(x)) > 0$ , then  $x_k > q_k$  and  $\eta_k > r_k$  for some  $i = k$ .

Lemma 2. If  $P(x) = F(x, \eta(x)) > 0$ , then  $x_i \leq q_i$  implies  $\eta_i(x) = 0$ .

Lemma 3 (Modified Gibbs' Lemma). Let  $f_i(x_i)$  be continuous with right- and left-derivatives. Let  $z = (z_1, \dots, z_n)$  maximize  $\sum_i f_i(x_i)$  constrained by  $\sum_i x_i = R_x > 0$  and  $x_i \geq s_i \geq 0$ . Then there exists a number  $\lambda$  such that

$$f'_i(z_i^+) \leq \lambda \quad \text{for all } i;$$

$$f'_i(z_i^-) \geq \lambda \quad \text{for all } i \text{ with } z_i > s_i.$$

Furthermore,  $\lambda$  is unique if  $f_i(x_i)$  is differentiable at  $x_i = z_i > s_i$  for some  $i$ .

Lemma 4. Either  $w_1 = 0$  or  $w_1 > r_1$ .

This lemma says that payment of a fraction  $f$ ,  $0 < f < 1$ , of a price of admission for the  $i$ th system is nonoptimal. The  $y$ -player's  $i$ th system cannot exist unless the price of admission,  $r_1$ , has been fully paid. Therefore an intermediate payment diminishes the  $y$ -player's resources without effecting the  $x$ -player. The corresponding result is obtained for the  $x$ -player as a corollary to the following theorem.

Theorem 5. Let us suppose, for each  $i$ , that  $g_i(x_i, y_i)$  is a strictly increasing function of  $x_i$  over the domain  $0 \leq x_i \leq D$ . The residual value  $V$  of (1.1)-(1.5) is a strictly increasing function of  $R_x$  on  $(q^*, D]$  where  $q^* \equiv \min_i q_i$ .

Proof. Suppose that  $R_x^* < R_x$ ,  $F(u, w) = V$  and  $F(u^*, w^*) = V^*$ . Assume that  $V(R_x^*) \geq V(R_x)$ . Select  $\xi$  as any point such that  $\sum_i \xi_i = R_x$  and  $\xi_i > u_i^*$  for each  $i$ . Defining  $\eta$  by:

$$F(\xi, \eta) = \min_y F(\xi, y) = P(\xi)$$

it is noted that

$$F(\xi, \eta) \leq F(u, w) \leq F(u^*, w^*).$$

From Lemma 1,  $u_i^* > q_i$  and consequently  $\xi_i > q_i$  for some  $i$ . By the increasing nature of  $g_i(x_i, y_i)$  for every  $i$ ,

$$F(u^*, \eta) < F(\xi, \eta) \leq F(u^*, w^*)$$

contrary to the definition of  $w^*$ . Hence  $V(R_x^*) > V(R_x)$ .

Corollary 6. Under the hypotheses of Theorem 5, either  $u_i = 0$  or  $u_i > q_i$ .

Proof. Define  $I \equiv \{i : 0 < u_i \leq q_i\}$ . If  $I$  is nonempty then  $z \equiv \sum_i u_i > 0$ .

Define  $u^*$  by  $u_i^* = u_i$  ( $i \notin I$ ) and  $u_i^* = 0$  ( $i \in I$ ). Then

$$V(R_x) = P(u) = P(u^*) \leq V(R_x - z).$$

Since  $R_x > q^*$ ,  $0 < V(R_x)$ . If  $R_x - z > q^*$  we have a contradiction to Theorem 5 and if  $R_x - z \leq q^*$ ,  $V(R_x - z) = 0$  so again we have a contradiction.

After introducing some notation two lemmas are presented; these lemmas are used to reduce the solution of (1.1)-(1.5) to the solution of a zero-sum, two-person continuous game (with pure strategy).

Notation. Let  $(u, w)$  be a solution of (1.1)-(1.5) and define:

$$A \equiv \{i : u_i > s_i\} \cap \{i : w_i > t_i\};$$

$$B \equiv \{i : u_i = s_i\} \cap \{i : w_i = t_i\};$$

$$C \equiv \{i : u_i > s_i\} \cap \{i : w_i = t_i\};$$

$$D \equiv \{i : u_i = s_i > 0\} \cap \{i : w_i > t_i\};$$

$$X \equiv \{x = (x_1, \dots, x_n) : x_i \geq \max(s_i, q_i) \text{ for } i \in A \cup C,$$

$$x_i = s_i \text{ for } i \in B \cup D \text{ and } \sum_{A \cup C} x_i = R_x - \sum_{B \cup D} s_i\};$$

$$Y \equiv \{y = (y_1, \dots, y_n) : y_i \geq \max(t_i, r_i) \text{ for } i \in A \cup D,$$

$$y_i = t_i \text{ for } i \in B \cup C \text{ and } \sum_{A \cup D} y_i = R_y - \sum_{B \cup C} t_i\}.$$

Define the game:

$$(2.1) \text{ Given: } G_{XY}(x, y) = \sum_A g_i(x_i, y_i) + \sum_B f_i(s_i, t_i) + \sum_C g_i(x_i, t_i) + \sum_D g_i(s_i, y_i)$$

$$(2.2) \text{ Constrained by: } x \in X, y \in Y$$

$$(2.3) \text{ Determine: } V_{XY} = \max_X P_{XY}(x) = \max_X \min_Y G_{XY}(x, y).$$

In (2.1), replace  $g_i(x_i, t_i)$  by  $h_i(x_i)$  whenever  $t_i = 0$ .

Lemma 7.  $P_{XY}(x)$  is a concave function of  $x$  in  $X$ .

This lemma is given in greater generality by Shiffman [4]; because of the inaccessibility of that reference the proof is given below.

Proof. Let  $x, x^* \in X$  and  $0 \leq \alpha \leq 1$ . Then  $P_{XY}[\alpha x + (1-\alpha)x^*] = \min_Y G_{XY}[\alpha x + (1-\alpha)x^*, y] \geq \min_Y [\alpha G_{XY}(x, y) + (1-\alpha)G_{XY}(x^*, y)] \geq \alpha P_{XY}(x, y) + (1-\alpha)P_{XY}(x^*, y).$

Hitherto there has been no restrictions related to the differentiability of  $g_i(x_i, y_i)$  and  $h_i(x_i)$ . For the remainder of this section we assume for each  $i$  that  $h'_i(x_i)$  exists and is continuous, and we assume that  $\partial g_i(x_i, y_i) / \partial x_i$  exists and is continuous in its variables taken together. The following lemma can be proved by appealing to the definition of the directional derivative and modifying slightly

the proof of Danskin's corresponding theorem [1, p. 19-22].

**Lemma 8.** Let  $\Gamma$  be a hypercurve in  $X$  and assume that  $X$  is not a single point.

Let  $D_\Gamma$  denote the directional derivative along  $\Gamma$ . For each  $x \in \Gamma$ ,  $D_\Gamma P(x)$  and  $D_\Gamma P_{XY}(x)$  exist.

**Theorem 9.** The strategies  $(u, w)$  are optimal strategies for the game (2.1)-(2.3).

**Proof.** Suppose to the contrary that  $(u, w)$  is not optimal for (2.1)-(2.3) and let  $(u^*, w^*)$  be a solution of (2.1)-(2.3).

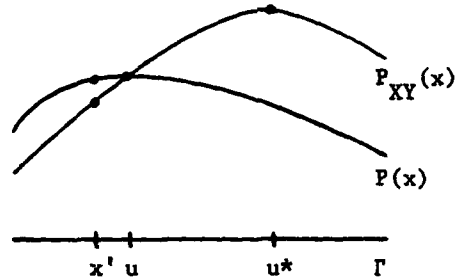


Figure 1. Typical Security Functions

Define  $\Gamma$  as the hyperline segment formed by

the intersection of  $X$  and the hyperline passing through  $u$  and  $u^*$ . Either  $X$  consists of one point, in which case  $u = u^*$ , or  $u$  lies in the interior of  $\Gamma$ . Since  $P(x)$  is maximized at  $x = u$  and  $D_\Gamma[P(u)]$  exists,  $D_\Gamma[P(u)] = 0$ . Since  $P_{XY}(x)$  is increasing as  $\Gamma$  is traversed from  $u$  to  $u^*$ ,  $D_\Gamma[P_{XY}(u)] > 0$ . This means that  $P_{XY}(x') < P(x')$  for some point  $x'$  such that  $u \in (x', u^*)$ , as illustrated in Figure 1. This conclusion contradicts the fact that  $P(x')$  is minimum over a space which includes  $Y$ . Hence  $u = u^*$ . Define  $\eta^*(x)$  by  $F(x, \eta^*(x)) = P_{XY}(x)$ . Since  $\eta^*(x)$  is well-defined on  $Y$  [cf. 4],  $w^* = w$ .

Although Theorem 9 reduces the solution of (1.1)-(1.5) to the solution of a game (2.1)-(2.3), the spaces  $X$  and  $Y$  (or equivalently the sets  $A, B, C, D$ ) are not *a priori* known. After establishing further notation a method for determining  $X$  and  $Y$  is provided.

Let  $\Pi = (A', B', C', D')$  be a partition of  $\{1, 2, \dots, n\}$  constrained by " $i \in D'$  implies  $s_1 > 0$ ." Define  $X'$  and  $Y'$  with respect to  $\Pi$  in the same manner as  $X$  and  $Y$  are defined with respect to  $A, B, C$ , and  $D$ . Let  $\Phi$  be the set of all such pairs  $(X', Y')$ . Finally define  $\Psi(X') \equiv \{Y' : (X', Y') \in \Phi\}$ .

Lemma 1C. For each  $(X', Y') \in \Phi$  let  $V_{X', Y'}$  be the value of the game (2.1)-(2.3) with  $(X, Y)$  replaced by  $(X', Y')$ . Define

$$V_{X'} \equiv \min_{Y' \in \Psi(X')} V_{X', Y'}.$$

Then for some  $(X, Y) \in \Phi$ ,

$$V = V_{XY} = V_{X'}.$$

Proof. By Theorem 9,  $V = V_{XY}$  for some  $(X, Y) \in \Phi$ . Suppose that  $V_{XY} > V_{X'} = V_{XY'}$ . Then

$$(2.4) \quad P(u) = P_{XY}(u) > P_{XY'}(u') \geq P_{XY'}(u)$$

where  $u'$  is the  $x$ -player's optimal strategy for the  $(X, Y')$ -game. The inequality (2.4) is inconsistent with the definition of  $P(u)$ . Hence  $V_{XY} = V_{X'} = V$ .

For the case of no committed resources it is shown in [3] for the example (1.6) that  $X = Y$ . Unfortunately, there does not appear to be a similar result for the committed resources problem. The following crude procedure shows how (1.1)-(1.5) can be solved.

Algorithm 11. (i) For each  $(X', Y') \in \Phi$  determine  $V_{X', Y'} = P_{X', Y'}(u_{X', Y'})$ .

(ii) Find  $V_{X'}$  for each  $X'$ .

(iii) Define  $U \equiv \{u_{X', Y'} : P_{X', Y'}(u_{X', Y'}) = V_{X'} = P(u_{X', Y'})\}$ .

(iv) Find  $P(u) = \max \{P(u_{X', Y'}) : u_{X', Y'} \in U\}$ .

(v) Determine  $u = u_{X', Y'}$ ,  $w = w_{X', Y'}$  and  $V = V_{X'}$  from step (iv).

It is noted that  $U$  is nonempty and whenever  $P_{X', Y'}(u_{X', Y'}) \neq P(u_{X', Y'})$ , this choice of  $X'$  could not have been optimal.

### III. OBSERVATIONS AND CONCLUSIONS

There has been no discussion in the preceding sections on how  $V_{X', Y'}$ ,  $u_{X', Y'}$  and  $w_{X', Y'}$  are determined. By imposing the additional hypothesis that  $\partial g_1(x_1, y_1)/\partial y_1$  exists and is decreasing in  $x_1$  for each fixed  $y_1$ , the "Gibbs' lemma approach"

of Danskin [1] can be used. The necessary generalizations for the application of this technique are straightforward. A class of functions which satisfy all of preceding hypotheses is:

$$F(x,y) = \sum_{i=1}^n \left[ \sum_{j=1}^{m_i} \alpha_{ij}(x_i) \beta_{ij}(y_j) \right]$$

where  $\alpha_{ij}$  and  $\beta_{ij}$  are respectively concave-increasing and convex-decreasing for  $x_i > q_i$  and  $y_j > r_j$ . Set  $\alpha_{ij}(x_i) = 0$  whenever  $0 \leq x_i \leq q_i$  and  $\beta_{ij}(y_j) = 0$  whenever  $0 \leq y_j \leq r_j$ . An example of a concave-increasing, convex-decreasing function which does not satisfy the above hypothesis is:

$$g(x,y) = (xy)^{-1} + 2 \ln x.$$

If the x-player has previously committed resources of  $v$  of  $n$  systems, at most  $3^{n-v} 4^v$  games must be solved. Although this bound grows quite rapidly, in practice  $n$  is small. For example, if  $n = 9$  and  $v = 5$  there are 82944 games to be solved compared to 19683 games for the corresponding problem with no committed resources ( $v = 0$ ). At the rate of 0.05 seconds of computer time per game, an upper bound of over an hour of computer time per choice of parameters is obtained. This illustrates the need for both careful programming and for a consideration of additional special properties a particular problem may possess. To consider a large number of systems there is a need for a more direct method of solving (1.1)-(1.5).

There are several other problems related to (1.1)-(1.5); for example, suppose that some of the  $g_i(x_i, y_i)$  were convex-convex. This additional complication may be treated by showing that additional investment should be made in at most one of the convex-convex systems. The analysis can then proceed by using the approach of Danskin [1, p. 52+]. Functions  $g_i(x_i, y_i)$  which are concave-convex for some uncomplicated regions of  $x_i$  and convex-convex elsewhere can also be considered without excessive difficulty. Of course, the additional complexities increase the computer time.

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Additional improvements in mathematical modeling are also needed. We end this work by posing two open questions. How can operational costs be separated from procurement costs? How does the time phasing of procurements affect the result?

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